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The van Kampen Expansion for the Fokker–Planck Equation of a Duffing Oscillator Excited by Colored Noise

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In Rodríguez and van Kampen's 1976 paper a method of extracting information from the Fokker–Planck equation without having to solve the equation is outlined. The Fokker–Planck equation for a Duffing oscillator excited by white noise is expanded about the intensity α of the forcing function. In Weinstein and Benaroya, the effect of the order of expansion is investigated by carrying the expansion to a higher order. The effect of varying the system parameters is also investigated. All results are verified by comparison to Monte Carlo experiments. In this paper, the van Kampen expansion is modified and applied to the case of a Duffing oscillator excited by colored noise. The effect of the correlation time is investigated. Again the results are compared to those of Monte Carlo experiments. It is found that the expansion compares closely with those of the Monte Carlo experiments as the correlation time τ_c is varied from 0.001 to 10 sec. Examination of the results reveals that the colored noise can be categorized in one of four ways: (1) for $\tau_c < \mathcal{O}(0.01\text{ sec})$ the noise can be considered as white for all intents and purposes, (2) for $\tau_c = \mathcal{O}(0.1\text{ sec})$ the noise can be considered white for some purposes, (3) for $\tau_c = \mathcal{O}(1.0\text{ sec})$ the correlated nature of the noise must be considered in an analysis, and (4) for $\mathcal{O}(1.0\text{ sec}) < \tau_c$ the noise can be considered as deterministic.

KEY WORDS: Fokker–Planck equation; white noise; colored noise; van Kampen expansion; Monte Carlo; Duffing oscillator.

1. INTRODUCTION

The Fokker–Planck equation has proven to be a useful tool in the analysis of simple nonlinear oscillators excited by stochastic processes. As a partial

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differential equation for the probability density function of the response, its solution completely defines the solution of the problem. It can be used to analyze both a single oscillator of the form

$$m\ddot{x} + \gamma(\dot{x}, x)\dot{x} + k(\dot{x}, x)x = \mathscr{F}(t) \tag{1}$$

or a system of multiple, linked oscillators of the form

$$\mathbf{M}\ddot{\underline{x}} + \mathbf{\Gamma}(\dot{\underline{x}}, \underline{x})\dot{\underline{x}} + \mathbf{K}(\dot{\underline{x}}, \underline{x})\underline{x} = \underline{\mathscr{F}}(t) \tag{2}$$

In many cases, a physical system can be approximated by such a system of nonlinear oscillators. The systems so modeled can range from a Brownian particle to structures excited by von Kármán vortex shedding. Such modeling can be useful for gaining insight into a problem and the way in which the system will behave as certain parameters are varied.

Once one has decided on the system of oscillators to be used to represent the physical system, the derivation of the Fokker–Planck equation is relatively straightforward, although tedious. The problem remains of how to solve it for the probability distribution of the response. In a very few cases, the Fokker–Planck equation can be solved analytically, but in most cases no analytical solution exists and one usually must resort to a numerical solution. However, this can be computationally intensive and gives little insight into the larger problem.

In their 1976 paper, Rodríguez and van Kampen[1] outline a method of dealing with the case of an oscillator excited by weak Gaussian white noise. The Fokker–Planck equation of the system is expanded about the intensity α of the driving function. This expansion is carried to $\mathcal{O}(\alpha^{1/2})$. In this way the statistics of the fluctuations are obtained directly. This method shows promise as a way to use the Fokker–Planck equation to gain useful information about a wider variety of systems than was possible before.

This is the second of a planned series of papers exploring the usefulness of this method. In the previous paper[2] the method was applied to the problem of a Duffing oscillator excited by Gaussian white noise. The inherent assumptions of the method were explained there in detail, the expansion was carried both to the same order as in the orginal paper and to $\mathcal{O}(\alpha^{3/2})$, and results were presented and compared to Monte Carlo experiments. Parametric studies were also performed on the parameter of expansion as well as on the other important variable in the expansion: the coefficient of damping.

In this paper, the expansion is applied to the case of a Duffing oscillator excited by exponentially correlated noise. A parametric study is performed on the correlation time τ_c and the results are compared to those of Monte Carlo experiments. The range of τ_c for which the correlated noise can be treated as white is identified as well as the range of τ_c for which the correlated nature of the noise cannot be ignored.

2. EXPANSION OF THE FOKKER–PLANCK EQUATION FOR A DUFFING OSCILLATOR EXCITED BY CORRELATED NOISE

In Gang[3] a method is outlined for deriving the Fokker–Planck equation of a system driven by colored noise. The procedure outlined consists of defining a dummy variable y to represent the colored noise process. y is defined as a first-order, linear, differential filter of the white noise process:

$$c_1 \dot{y}(t) = c_2 y(t) + c_3 W(t) \tag{3}$$

where c_i are constants and $W(t)$ is the Gaussian white noise process defined by

$$\begin{aligned} \langle W(t) \rangle &= 0 \\ \langle W(t)\, W(t') \rangle &= 2\alpha\delta(t-t') \end{aligned} \tag{4}$$

The system is now formulated as a system of first-order differential equations driven by $y(t)$:

$$\dot{y} = \frac{c_2}{c_1} y + \frac{c_3}{c_1} W \tag{5}$$

$$\dot{x} = F_x(x, v, y) \tag{6}$$

$$\dot{v} = F_v(x, v, y) \tag{7}$$

Several examples of colored noise filters exist in the literature. Billah and Shinozuka[4] use

$$\tau_c \dot{y}(t) = -y(t) + W(t) \tag{8}$$

where τ_c is the correlation time. It can be seen that as $\tau_c \to 0$, $y(t) \to W(t)$. For this derivation, it will be assumed that τ_c is not vanishingly small, so that Eq. (8) can be written as

$$\dot{y}(t) = -\mu y(t) + \mu W(t) \tag{9}$$

where $\mu = 1/\tau_c$.

The Duffing oscillator excited by colored noise can now be written as

$$\ddot{x}(t) + \gamma \dot{x}(t) + x(t) + x^3(t) = y(t) \tag{10}$$

where γ is the coefficient of damping.

By defining $v(t) \equiv \dot{x}(t)$, one can recast the system as three linked, ordinary differential equations in time in the form of Eqs. (5)–(7):

$$\dot{x}(t) = v(t) \tag{11}$$

$$\dot{v}(t) = y(t) - \gamma v(t) - x(t) - x^3(t) \tag{12}$$

$$\dot{y}(t) = -\mu y(t) + \mu W(t) \tag{13}$$

Define $f(x, v, y; t)$ to be the joint probability density function of $x(t)$, $v(t)$, and $y(t)$ at time t. The Fokker–Planck equation for $f(x, v, y; t)$ can now be derived as

$$\frac{\partial f}{\partial t} = -v \frac{\partial}{\partial x} f - \frac{\partial}{\partial v} [(y - \gamma v - x - x^3) f] + \mu \frac{\partial}{\partial y} (yf) + \mu^2 \alpha \frac{\partial^2}{\partial y^2} f \tag{14}$$

Equation (14) is the governing equation for the time evolution of the transition probability density function $f(x, v, y; t)$. From this point the derivation of the previous paper[2] will be followed.

As was shown in the previous paper, the response of the oscillator can be separated into a deterministic component due to the initial conditions and a random component of magnitude $\mathcal{O}(\sqrt{\alpha})$. However, by assuming that the oscillator is initially at rest, the deterministic component can be shown to be equal to zero. Therefore the following substitutions are made into Eq. (14):

$$x = \sqrt{\alpha}\, \zeta \tag{15}$$

$$v = \sqrt{\alpha}\, \eta \tag{16}$$

$$y = \sqrt{\alpha}\, \rho \tag{17}$$

$$f(\sqrt{\alpha}\, \zeta, \sqrt{\alpha}\, \eta, \sqrt{\alpha}\, \rho; t) = \alpha^{-3/2} \Pi(\zeta, \eta, \rho; t) \tag{18}$$

The factor $\alpha^{-3/2}$ will be omitted from the definition of $\Pi(\zeta, \eta, \rho; t)$. If carried through the derivations, it would be divided out at a later stage of the derivation.

It must be explained why, as is implied in Eq. (17), y is of the order of magnitude $\alpha^{1/2}$. This is because y is of the same order of magnitude as the white noise forcing function F. But it can be seen from the definition of F, Eqs. (4), that $\alpha = 2\sigma_F^2$. Therefore F and consequently y are of $\mathcal{O}(\alpha^{1/2})$.

The relationships between the partial derivatives of f and Π are obtained as

$$\alpha^{1/2} \frac{\partial f}{\partial x} = \frac{\partial \Pi}{\partial \zeta} \tag{19}$$

$$\alpha^{1/2}\frac{\partial f}{\partial v}=\frac{\partial \Pi}{\partial \eta} \tag{20}$$

$$\alpha^{1/2}\frac{\partial f}{\partial y}=\frac{\partial \Pi}{\partial \rho} \tag{21}$$

$$\frac{\partial f}{\partial t}=\frac{\partial \Pi}{\partial t} \tag{22}$$

This yields the following transformed Fokker–Planck equation:

$$\frac{\partial}{\partial t}\Pi = -\eta\frac{\partial}{\partial \zeta}\Pi - \frac{\partial}{\partial \eta}[(\rho-\gamma\eta-\zeta-\alpha\zeta^3)\Pi] + \mu\frac{\partial}{\partial \rho}(\rho\Pi) + \mu^2\frac{\partial^2}{\partial \rho^2}\Pi \tag{23}$$

By manipulating the left-hand side of Eq. (23), one obtains

$$-\frac{\partial \Pi}{\partial \zeta}\dot{\zeta} - \frac{\partial \Pi}{\partial \eta}\dot{\eta} - \frac{\partial \Pi}{\partial \rho}\dot{\rho} = -\eta\frac{\partial}{\partial \zeta}\Pi - \frac{\partial}{\partial \eta}[(\rho-\gamma\eta-\zeta-\alpha\zeta^3)\Pi] + \mu\frac{\partial}{\partial \rho}(\rho\Pi) + \mu^2\frac{\partial^2}{\partial \rho^2}\Pi \tag{24}$$

If one substitutes the definitions of ζ, η, and ρ into Eqs. (11)–(13), one can use the resultant equations to separate Eq. (24) into the following three equations:

$$\dot{\zeta}\Pi = \eta\Pi \tag{25}$$

$$\dot{\eta}\Pi = \{2\rho-\gamma\eta-\zeta-\alpha\zeta^3\}\Pi \tag{26}$$

$$\dot{\rho}\Pi = -\mu\rho\Pi - \mu^2\Pi_\rho \tag{27}$$

Using Eqs. (27), (25), and (26), one can find the time derivatives of the higher-order moments of ζ, η, and ρ. As an example, the time derivative of $\langle \rho^2 \rangle$ will derive. To derive $(d/dt)\langle \rho^2 \rangle$, one multiplies Eq. (27) by 2ρ, notes that $2\rho\dot{\rho} = (d/dt)(\rho^2)$, and gets

$$\rho^2\Pi = -\mu\rho^2\Pi - \mu^2\rho\Pi_\rho \tag{28}$$

Integrating this over the range of all three arguments of Π and noting the definition of expectation, one finds

$$\frac{d}{dt}\langle \rho^2 \rangle = -2\mu\langle \rho^2 \rangle - 2\mu^2\iiint_{-\infty}^{\infty}\rho\Pi_\rho\, d\rho\, d\zeta\, d\eta \tag{29}$$

The second term on the right-hand side can be expanded as follows:

$$\iiint_{-\infty}^{\infty} \rho \Pi_\rho \, d\rho \, d\zeta \, d\eta = \iiint_{-\infty}^{\infty} [(\rho\Pi)_\rho - \Pi] \, d\rho \, d\zeta \, d\eta$$

$$= -1 + \iint_{-\infty}^{\infty} [\rho\Pi]_{\rho=-\infty}^{\rho=\infty} = -1 \tag{30}$$

This last step is possible because the existence of $\langle \rho \rangle$ guarantees that $\rho\Pi \to 0$ as $\rho \to \pm\infty$. Equation (29) becomes

$$\frac{d}{dt}\langle \rho^2 \rangle = -2\mu\langle \rho^2 \rangle + 2\mu^2 \tag{31}$$

The time derivatives of the other second-order moments are similarly obtained:

$$\frac{d}{dt}\langle \zeta^2 \rangle = 2\langle \zeta\dot{\zeta} \rangle = -2\langle \zeta\eta \rangle \tag{32}$$

$$\frac{d}{dt}\langle \eta^2 \rangle = 2\langle \eta\dot{\eta} \rangle = -2\langle \zeta\eta \rangle - 2\gamma\langle \zeta^2 \rangle + 4\langle \eta\rho \rangle - 2\alpha\langle \zeta^3\eta \rangle \tag{33}$$

$$\frac{d}{dt}\langle \zeta\eta \rangle = \langle \dot{\zeta}\eta \rangle + \langle \zeta\dot{\eta} \rangle = -\langle \zeta^2 \rangle - \gamma\langle \zeta\eta \rangle - \langle \eta^2 \rangle + 2\langle \zeta\rho \rangle - \alpha\langle \zeta^4 \rangle \tag{34}$$

$$\frac{d}{dt}\langle \rho^2 \rangle = -2\mu\langle \rho^2 \rangle - 2\mu^2 \tag{35}$$

$$\frac{d}{dt}\langle \zeta\rho \rangle = -\mu\langle \zeta\rho \rangle - \langle \eta\rho \rangle \tag{36}$$

$$\frac{d}{dt}\langle \eta\rho \rangle = -\langle \zeta\rho \rangle - \gamma\langle \eta\rho \rangle + 2\langle \rho^2 \rangle - \alpha\langle \zeta^3\rho \rangle \tag{37}$$

3. RESULTS AND FIGURES

Figures 1, 3, and 5 show the results of the expansion for $\alpha = 0.1$, $\gamma = 1.0$, and different values of the correlation time τ_c: $\tau_c = 0.001$, 0.01, 0.1, 1.0, 10.0. Figures 2, 4, and 6 show the results of Monte Carlo experiments for similar cases.

As in the previous paper, there is very good agreement between the two methods, with similar differences as well. The analytical methods consistently give results slightly greater in magnitude than do the Monte Carlo experiments. This phenomenon was seen in the previous paper. The

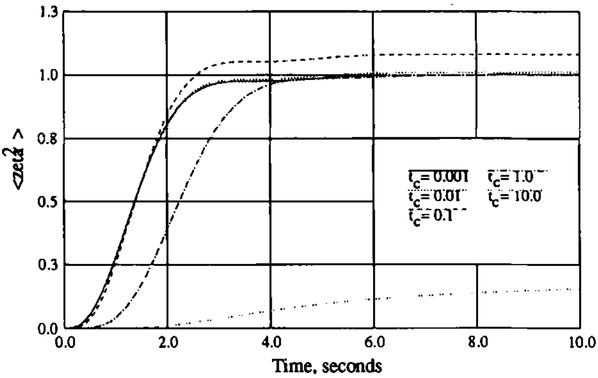

Fig. 1. $\langle\zeta^2\rangle$ versus time for different values of τ_c, $\gamma=1.0, \alpha=0.1$, calculated analytically to $\mathcal{O}(\alpha^{3/2})$.

analytical results are also consistently smoother, showing none of the small-time scale fluctuations inherent in the Monte Carlo technique.

It can be seen in all the curves that the traces representing $\tau_c=0.001$ and $\tau_c=0.01$ are almost identical. In some cases, most notably Figs. 1 and 3, the two traces are almost indistinguishable. The difference between the traces representing $\tau_c=0.001$ and $\tau_c=0.01$ are slightly more pronounced in the Monte Carlo results. However, the difference between the $\tau_c=0.001$ and $\tau_c=0.01$ traces of the Monte Carlo results is only of the order of the small-time-scale fluctuations inherent in the Monte Carlo technique. These figures indicate that for τ_c of $\mathcal{O}(0.01)$ or less, the noise can be assumed to

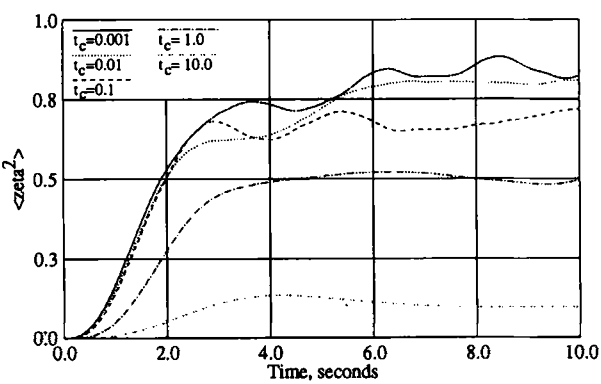

Fig. 2. $\langle\zeta^2\rangle$ versus time for different values of τ_c, $\gamma=1.0, \alpha=0.1$, calculated by Monte Carlo simulation.

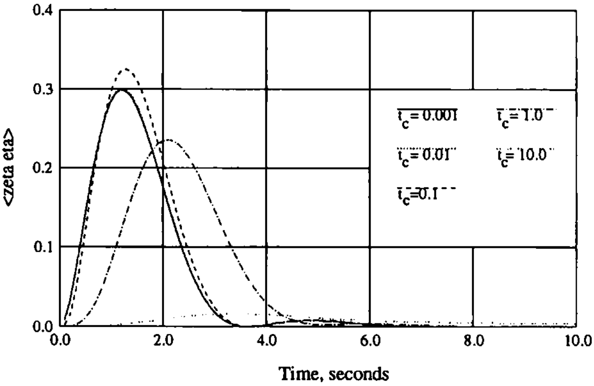

Fig. 3. $\langle \zeta\eta \rangle$ versus time for different values of τ_c, $\gamma = 1.0$, $\alpha = 0.1$, calculated analytically to $\mathcal{O}(\alpha^{3/2})$.

be uncorrelated, or white. The traces representing the response for $\tau_c = 0.1$ differ noticeably from those representing the results for $\tau_c < 0.1$. However, even for $\tau_c = 0.1$, the results are still quite close to those for $\tau_c < 0.1$ and the white noise approximation may still be useful for some uses. One would expect the effects of the correlated nature of the noise to become significant at about $\tau_c = 0.1$: at $\tau_c = 0.1$, the correlation time begins to become comparable to the natural period of the oscillator, which is about 1 sec. The correlated nature of the noise appears as an effect of time scale τ_c on the time history of the correlated noise. If the time scale of the correlation is much smaller than the natural period of the oscillator, then the oscillator

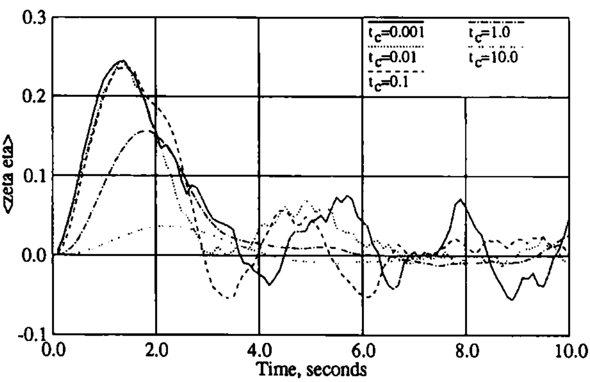

Fig. 4. $\langle \zeta\eta \rangle$ versus time for different values of τ_c, $\gamma = 1.0$, $\alpha = 0.1$, calculated by Monte Carlo simulation.

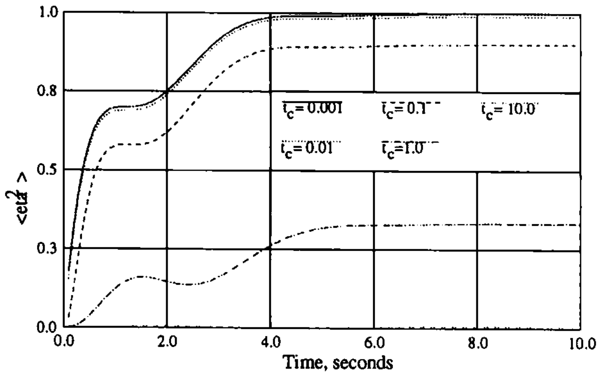

Fig. 5. $\langle\eta^2\rangle$ versus time for different values of τ_c, $\gamma = 1.0$, $\alpha = 0.1$, calculated analytically to $\mathcal{O}(\alpha^{3/2})$.

cannot respond to this effect. However, as τ_c approaches the natural period of the oscillator, the oscillator can be, and is, affected.

It can likewise be seen that as the correlation time of the noise becomes much greater than the natural period of the oscillator, the magnitude of the random response approaches zero. This is because when the correlation time is much larger than the natural period of the oscillator, the oscillator responds to the noise as if it were deterministic. Hence the figures show little random response for $\tau_c = 1.0$ sec and almost none at all for $\tau_c = 10.0$ sec.

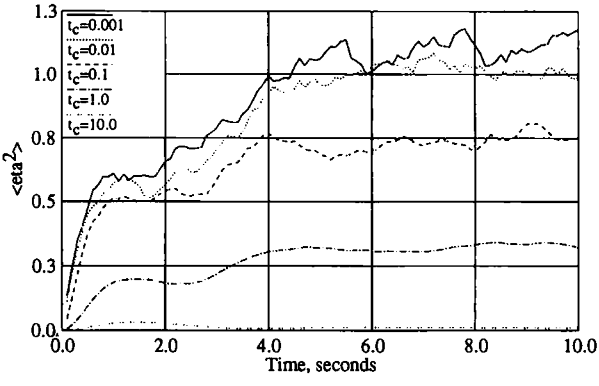

Fig. 6. $\langle\eta^2\rangle$ versus time for different values of τ_c, $\gamma = 1.0$, $\alpha = 0.1$, calculated by Monte Carlo simulation.

4. CONCLUSIONS

The overwhelming similarity between the results given by the two methods implies that this adaptation of the van Kampen expansion is an accurate tool for predicting the statistics of the response of an oscillator excited by colored noise. However, it was also seen that, depending on the magnitude of the correlation time as compared to the natural period of the oscillator, simplifying assumptions can be made that obviate the need for this adaptation.

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